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Letter to the Editor

A Note on "Approximation of Bounded Sets"

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In [2], Freilich and McLaughlin looked for a characterization of best simultaneous approximations (Chebyshev centers) for bounded (not necessarily compact) sets. They considered the following general setting:

Let X be a normed linear space, V and F nonempty subsets of X with V convex and F bounded. We are looking for $v_0 \in V$ which minimizes $r(v, F) = \sup_{x \in F} \|x - v\|$. Let K be a $\sigma(V, X^*)$-compact subset of the dual unit ball $B(X^*)$ which norms $F - V$, i.e., such that $\|x - v\| = \sup_{f \in K} f(x - v)$ for all $x \in F$, $v \in V$ (we may take, e.g., $K = B(X^*)$). On the compact space $(K, \sigma(V, K))$, consider the upper-semicontinuous function $U_F^+(f) = \limsup_{K \ni g \to_{\sigma(V,K)} f} \sup_{x \in F} g(x) = \inf \sup_{g \in W} \sup_{x \in F} g(x)$ over all $\sigma(V, K)$-neighborhoods W of f in K.

The main result of [2], Theorem 2, claims that if $v_0 \in V$, then v_0 is a best simultaneous approximation of F in V if and only if, for each $v \in V$, there is an $f \in \operatorname{ext} K$ satisfying:

(1) $U_F^+(f) - f(v_0) = r(v_0, F)$ and

(2) $f(v - v_0) \leqslant 0$.

The necessity part is, unfortunately, false as can be seen by taking $V = X = c_0$, $F = \{\pm((n-1)/n)\, e_n;\ n = 1, 2,...\}$ (where (e_n) is the unit vector basis of c_0), $K = B(c_0^*) = B(l_1)$. Clearly $v_0 = 0$ is a best simultaneous approximation of F, with $r(0, F) = 1$. However, no $f \in \operatorname{ext} K$ can satisfy (1): such an f must be of the type $\pm e_j^*$ for some j (where (e_j^*) is the unit vector basis of l_1), and if $\varepsilon > 0$ is any and $g \in W_\varepsilon = \{g \in K;\ |g(e_j) - f(e_j)| < \varepsilon\}$, then $\|g - f\| = \sum_{i \neq j} |g(e_i)| + |g(e_j) - f(e_j)| = \|g\| - |g(e_j)| + |g(e_j) - f(e_j)| < 1 - (1 - \varepsilon) + \varepsilon = 2\varepsilon$, so that $\sup_{x \in F} g(x) < \sup_{x \in F} f(x) + \|f - g\| \sup_{x \in F} \|f\| < 1 - (1/j) + 2\varepsilon$, thus $U_F^+(f) \leqslant 1 - (1/j)$.

The mistake is due to Lemma 1, which claims that U_F^+ is a convex function on K. Indeed, in our example we have $U_F^+(0) = 1$ while U_F^+

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$(\pm e_j) = 1 - (1/j)$. The error in the proof is the assertion that the translate of a $\sigma(V, K)$-open set is open again (in the relative topology).

The correct version of the theorem, using the same notation, seems to be:

THEOREM 1. *$v_0 \in V$ is a best simultaneous approximation of F in V if and only if for every $v \in V$ there is an f in the $\sigma(V, K)$-closure of* ext K *which satisfies* (1) *and* (2).

Proof. For the sufficiency part, any $f \in B(X^*)$ satisfying (1) and (2) will do: For, given $v \in V$ and any $\varepsilon > 0$, take $g \in K$ with $|g(v_0) - f(v_0)| < \varepsilon$, $|g(v) - f(v)| < \varepsilon$ and $x \in F$ with $g(x) - f(v_0) > r(v_0, F) - \varepsilon$. Then $\|x - v\| > g(x - v) = g(x) - g(v) > g(x) - \varepsilon - f(v) \geqslant g(x) - \varepsilon - f(v_0) > r(v_0, F) - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $r(v, F) = \sup\{\|x - v\| : x \in F\} \geqslant r(v_0, F)$.

For the necessity, observe first that $U_F^+(f) - f(v_0) \leqslant r(v_0, F)$ for all $f \in K$. Indeed, if $\varepsilon > 0$ is any and $W = \{g \in K;\ |f(v_0) - g(v_0)| < \varepsilon\}$, then for all $x \in F$ and $g \in W_\varepsilon$, we have $g(x) - f(v_0) < g(x) - g(v_0) + \varepsilon \leqslant \|x - v_0\| + \varepsilon \leqslant r(v_0, F) + \varepsilon$.

Given $v \in V$, take $x_n \in F$ with $\|x_n - v\| > r(v_0, F) - (1/n)$ and then take $f_n \in \text{ext}\, K$ with $f_n(x_n - v) > r(v_0, F) - (1/n)$. Let g_1 be a $\sigma(V, K)$-limit point of a subnet $(f_{n(\alpha)})$ of the (f_n). Then $U_F^+(g_1) - g_1(v) \geqslant \limsup_\alpha f_{n(\alpha)}(x_{n(\alpha)}) - \lim_\alpha f_{n(\alpha)}(v) = \limsup_\alpha f_{n(\alpha)}(x_{n(\alpha)} - v) \geqslant r(v_0, F)$. g_1 satisfies (2), since $g_1(v - v_0) = \lim_\alpha f_{n(\alpha)}(v - v_0) \leqslant \limsup_\alpha (f_{n(\alpha)}(x_{n(\alpha)} - v_0) - f_{n(\alpha)}(x_{n(\alpha)} - v)) \leqslant \limsup_\alpha f_{n(\alpha)}(x_{n(\alpha)} - v_0) - \liminf_\alpha f_{n(\alpha)}(x_{n(\alpha)} - v) \leqslant r(v_0, F) - r(v_0, F) = 0$.

Let $v_n = (1/n)\, v + (1 - (1/n))\, v_0 \in V$. Let $g_n \in \overline{\text{ext}}\, K$ be as above, i.e., with $U_F^+(g_n) - g_n(v_n) \geqslant r(v_0, F)$ and satisfying (2). Let f be a $\sigma(V, K)$-limit point of a subnet $(g_{n(\beta)})$ of the (g_n). Then f clearly satisfies (2). Since $U_F^+(g_n) - g_n(v_0) = U_F^+(g_n) - g_n(v_n) - (1/n)\, g_n(v - v_0) \geqslant r(v_0, F) - (1/n)\, g_n(v - v_0)$ and since $U_F^+ - v_0$ is upper-semicontinuous on K, we have $U_F^+(f) - f(v_0) \geqslant \liminf_n (r(v_0, F) - (1/n)\, g_n(v - v_0)) = r(v_0, F)$.

As in Proposition 2.7 of [1], one can proceed from Theorem 1 to prove the noncompact analogue of the Laurent–Tuan theorem:

THEOREM 2. *With the same assumptions and notations, $v_0 \in V$ is a best simultaneous approximation of F in V if and only if there is $\varphi_0 \in \overline{\text{conv}}\{f \in \overline{\text{ext}}\, K; f \text{ satisfies } (1)\}$ such that $\varphi_0(v_0) = \max_{v \in V} \varphi(v)$.*

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